InnoSpaceTool Unit 2 complementary material - Waves and Signals

Endurosat Team

This project has received funding from the European Unions Horizon 2020 research and innovation programme under grant agreement No 768049

In this section, we will familiarize ourselves with waves. First we will briefly discuss sinusoidal waves, which all readers should be familiar with from any physics classes at school and then we will get to the more general case. Communications rely heavily on electromagnetic waves, but studying waves is important on a more general scale, since it will give us a better understanding of concepts like spectra and the relations between the shape of a signal and its components in terms of frequencies.

1 Sinusoidal waves

Thinking of waves, most people imagine the familiar physical process of surface water waves or perhaps the waves of a guitar string. These two are examples of free waves and standing waves, and we will soon understand the difference between them, but let's give a basic definition of a wave first, and try to create one mathematically.
Def: *A wave is a periodic disturbance of a medium, which does NOT carry the medium itself with it, but which carries energy.*
This is an idealized definition and it is not fully formal, but it is sufficient for our purposes. Notice that the term "periodic" refers to repetitions both in time and space! Let us try to construct a simple 1-dimensional surface wave. An undisturbed surface can be seen simply as a constant function, then a wave will be some periodic function $U(x, t)$, which tells us how much the medium is displaced from its original position in every point and for every time. Now, consider a fixed position ($x = x_0 = const$), then for this position the function becomes simply $U(x_0, t) = U(t)$, schematically shown bellow.

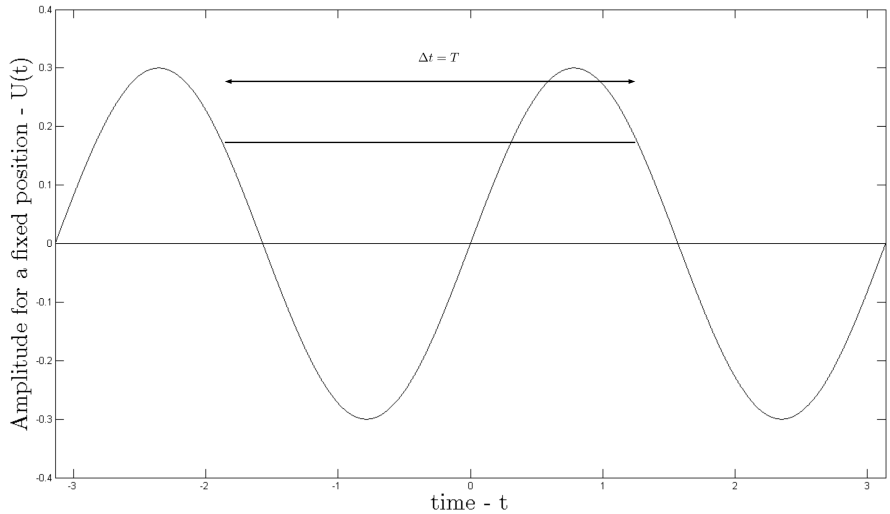

Fig.2.1 A periodic function of time with the period T indicated

Then it is obvious that $U(t+T) = U(t)$ from the periodicity condition, where T is the period of the wave (the shortest time which separates two crests or any two points at the same phase of the wave). So the amplitude a full period in the future will be the same as that at a given moment. Since the definition stated that periodicity must be present both in time and in space, for a fixed time ($t = const$), we must also have $U(x+\lambda) = U(x)$ for some "length period" denoted λ. This "length period" is called wavelength and is defined as the shortest distance (in space) between two crests or two points at the same phase (so exactly like the period, but in space). The simplest periodic function which everyone should be familiar with is the sine function. We can then try to construct our wave as a sine function of both space and time, with time period given by T and space period given by λ. Let's remind ourselves that the function $y = \sin x$ is a periodic function and its period is 2π, so $\sin(x \pm 2\pi) = \sin x$. To make the period of this function different (say T), we just need to form the fraction $\frac{2\pi}{T}$ and use it as a factor inside of the sine, since for this new function:

$$\sin\left(\frac{2\pi}{T}(x \pm T)\right) = \sin\left(\frac{2\pi x}{T} \pm 2\pi\right) = \sin\left(\frac{2\pi x}{T}\right)$$

Using this technique for both of our variables t and x, we can form two new constants from the period T and the wavelength λ:

$$\omega = \frac{2\pi}{T} \tag{1.1}$$

$$k = \frac{2\pi}{\lambda} \tag{1.2}$$

ω is known as the angular frequency, while k is called the wavenumber, for the given wave. In terms of these constants, then, we can write a periodic sine function of both time and space as

$$U(x,t) = \sin(\omega t \pm kx)$$

It is easy to see that this function has all the properties we need:

1. For $t = const$, the function is periodic in space with period λ
2. For $x = const$, the function is periodic in time with period T

In fact the relation gives 2 functions, one for the $+$ sign and one for the $-$ sign, which are both legitimate waves. The first one corresponds to a wave moving to the left (so in the $-x$ direction), and the second one corresponds to a wave moving to the right (so in the $+x$ direction). To see this, take $\sin(\omega t - kx)$. This can be written as $\sin(\omega(t - \frac{k}{\omega}x))$. Since ω is a constant, if we try to find all the points in space, which have the same phase, we just need to look for $t - \frac{k}{\omega}x = const$. However, we can find the following relation between ω and k:

$$\frac{k}{\omega} = \frac{T}{\lambda} = \frac{Time\ period}{Length\ period} = \frac{1}{c} \tag{1.3}$$

Where c is the speed of the wave since λ is the effective distance a point of it moves in space for a time T, and so λ/T has the dimensionality of speed (note that a point of the wave is moving, and not the matter itself). It follows, that $t - x/c = const$ gives us all the points with the same phase. Differentiating this expression with respect to time, we get:

$$1 - \frac{1}{c}\frac{dx}{dt} = 0$$

since the derivative of a constant is 0. It follows, that $dx/dt = c$ and so the wave's velocity is positive (so the wave is moving in the $+x$ direction). There are two types of velocities for waves - group velocity and phase velocity, but for the free sinusoidal waves the two match and so we will simply call this the wave velocity.
If we had taken the other solution ($\sin(\omega t + kx)$, we would have obtained $\frac{dx}{dt} = -c$ and so it corresponds to a wave with negative velocity (moving in the $-x$ direction) indeed.
To finalize our introduction to waves, we will remark that we have omitted the amplitude and the starting phase term so far. The amplitude is simply the factor in front of our sine function, which gives the maximum deviation from the undisturbed surface and for our current purposes, it is a constant. The starting phase is an additional term inside of the function, which tells us how disturbed the surface is in the beginning at the starting position. In the solutions we have considered so far, it is easy to see that for $t = 0$ and $x = 0$, we have simply $U(0,0) = \sin(0)$ and so our starting phase is 0. If we write $U(x,t) = \sin(\omega t \pm kx + \varphi_0)$, then for $x = t = 0$, we will have $U(0,0) = \sin(\varphi_0)$, which is generally not 0. This function is more general than our initial assumption as it allows us to "start" the wave at any of its points. It possesses all of the properties listed above, and so we are finally ready to give our general definition of a sinusoidal wave:
Def: *A free sinusoidal wave with period T, wavelength λ, amplitude A and initial phase φ_0, is a function of the form:*

$$U(x,t) = A\sin(\omega t \pm kx + \varphi_0) \tag{1.4}$$

where ω and k, given by 1.1 and 1.2 are the angular frequency and the wavenumber, and the signs $\pm$ correspond to a "right-to-left" and "left-to-right" wave motion respectively.
Note that this is a 1-dimensional wave (since it depends on just 1 coordinate). The generalization to higher-dimensional waves is quite easy as we will see later on.

The relation 1.3 we showed is quite important. Using the definition of k and ω and the fact that $\nu = 1/T$ is the standard frequency of a wave (in Hz), we can rewrite it as:

$$c = \frac{\omega}{k} = \frac{\lambda}{T} = \nu\lambda \tag{1.5}$$

This expression gives us the relation between the wavelength and the frequency of a free sinusoidal wave and it is valid for such waves in any media, making it quite general!
Example: Consider a free electromagnetic wave with frequency 2.4 GHz ($\nu = 2.4.10^9$ Herz). The speed of light in vacuum is about 300 000 km/s ($c = 3.10^8$m/s). Then:

$$\lambda = \frac{c}{\nu} = \frac{3.10^8}{2,4.10^9} = \frac{3}{2,4}.10^{-1} \cong 0.125\text{m}$$

This tells us that the wavelength of such a radio wave must be $0.125m = 12.5cm$, which we will later see to be in the S band.

2 General waves, a nice guess

Our definition 1.4 certainly has all the properties we would expect from a wave, but it is quite restrictive in the sense that our wave always has the same shape! For the current section, we will restrict ourselves to a fixed point in space ($x = const$) and so our wave will only depend on time - $U(t) = A\sin(\omega t + \varphi_0)$. We have included the constant term kx in the initial phase and so this is now simply a time-dependant oscillation of some quantity U (for example this would be the output voltage function of an AC electrical generator). Modifying the amplitude A, the phase φ_0 and the frequency ω would change the height, the starting point or the width of the sine function, but it will always remain a sine function. Surely it can't be the only periodic function which satisfies the requirements for a wave. The example bellow shows a periodic function which is clearly not a sine function and can also describe a wave.

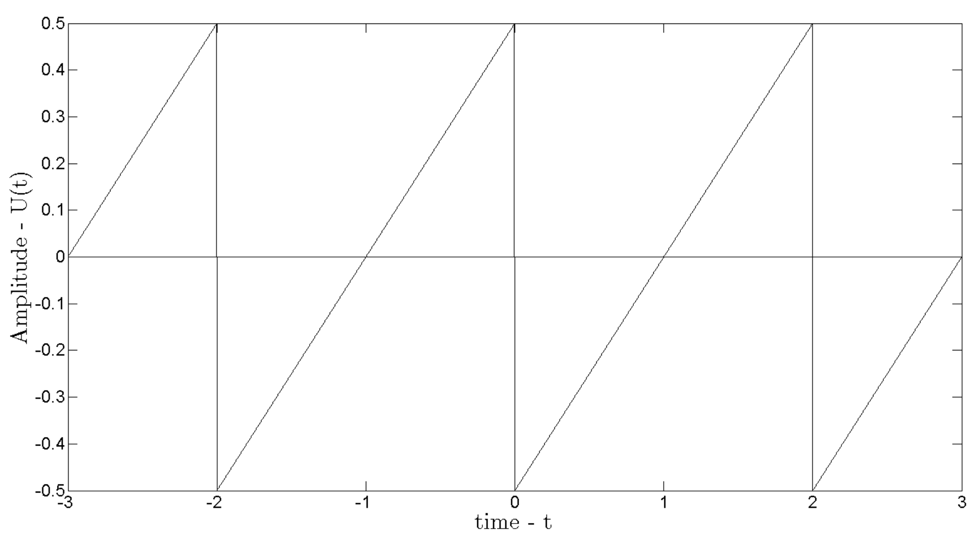

Fig.2.2 A periodic function which corresponds to $f(t) = \frac{1}{2}(x-1)$ in the region $x \in [0,2]$ and has a period $T = 2$.

So how do we construct periodic functions in general? The simplest guess would be to add different versions of the only periodic function we have now and to try and "match" their parameters so they fit our desired function. The function we are trying to match can be written as $f(t) = \frac{1}{2}(x-1)$ in the region $x \in [0,2]$. Let's take the following "guess" combination of sine functions:

$$U_{guess}(t) = -\frac{1}{\pi}\sin(\pi t) - \frac{1}{2\pi}\sin(2\pi t)$$

Plotting them alongside our original function, we can see that there is a general resemblance, but they differ a lot in some points (Fig.2.3) .

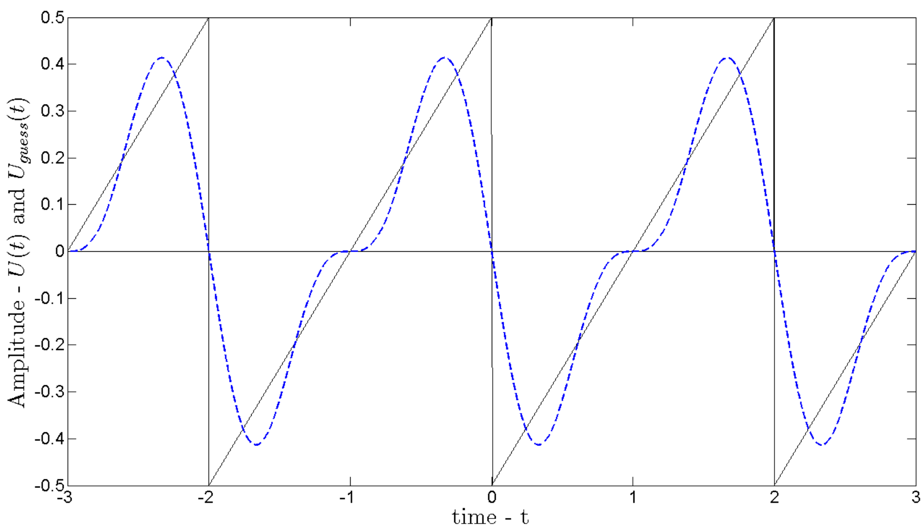

Fig.2.3 A comparison of U_{guess}(dashed blue line) and the real $U(t)$(solid black line).

We can try a better "guess" by adding more terms, another try might be:

$$U'_{guess}(t) = -\frac{1}{\pi}\sin(\pi t) - \frac{1}{2\pi}\sin(2\pi t) - \frac{1}{3\pi}\sin(3\pi t)$$

This new function is shown alongside the original bellow on (Fig.2.4), and we can definitely see some improvement, although we are not quite matching the function yet.

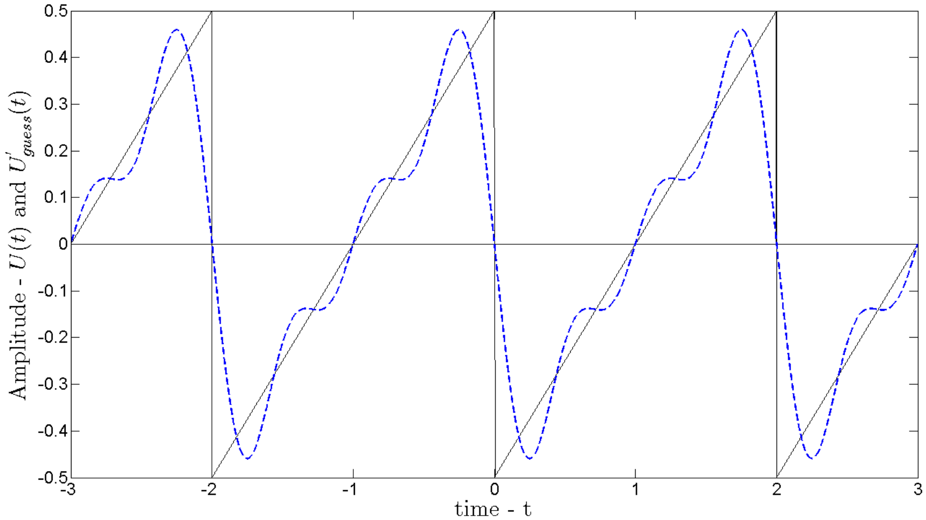

Fig.2.4 A comparison of U'_{guess}(dashed blue line) and the real $U(t)$(solid black line).

Looking more closely at the guess, we can see that it is a combination of sine functions with different angular frequencies (therefore different periods), each of which is a multiple of the first. Also, we can see that the amplitude of each sine seems to be the inverse of its angular frequency with a minus sign. We can therefore try to add more terms like these in $U'_{guess}(t)$, since they seem to add more accuracy, but when should we stop? The mathematical answer is never - we need to form an infinite sum to truly reach the desired function! In reality, however, we can stop whenever we have approximated our desired function closely enough, as no machine can ever add truly infinitely many terms in our sum. Since each term is of the form $-(1/(n\pi))\sin(n\pi t)$, our function will be perfectly described by the following series:

$$U(t) = \lim_{N \longrightarrow \infty} -\sum_{n=1}^{N} \frac{1}{n\pi} \sin(n\pi t) = -\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin(n\pi t)$$

On the graph bellow, you can see how this sum looks for $N = 5, N = 50$ and $N = 500$.

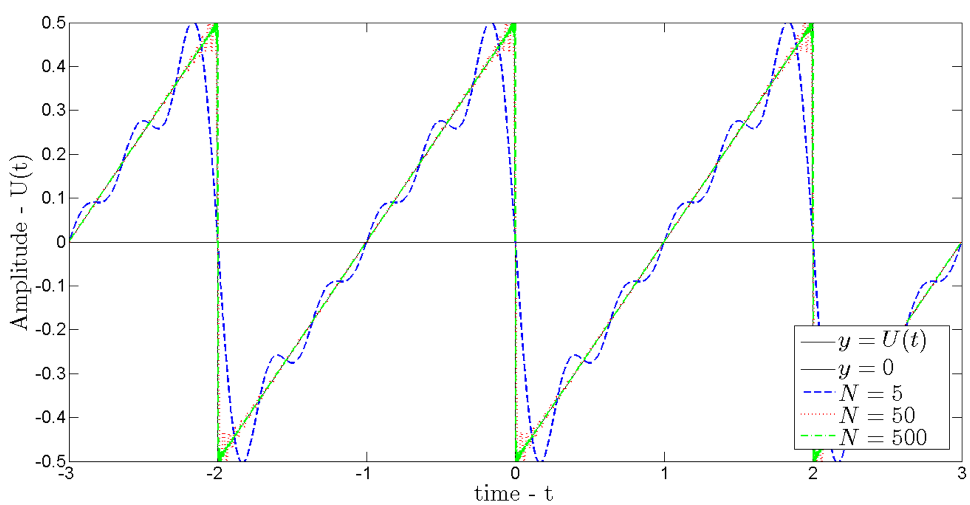

Fig.2.5 A comparison of $U(t)$ for $N = 5$ (dashed blue line), $N = 50$ (dotted red line) and $N = 500$ (dash-dotted green line).

For $N = 5$ we can still see large deviations everywhere (although the resemblance is quite clear). For $N = 50$, the only deviations left are near the sharp edges of the graph (this is true since our function is discontinuous at those points). For $N = 500$, it becomes really hard to see any notable deviations (Fig.2.6).

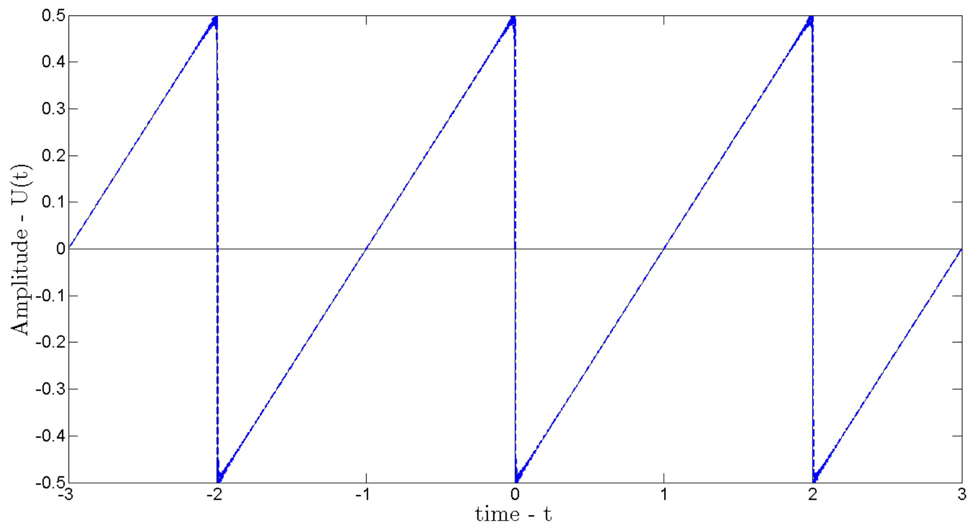

Fig.2.6 A comparison of $U(t)$ for $N = 500$ (dotted blue line)

Of course, if we are still not satisfied, we can take 5000 or 50000 terms in the constructed sum, but this will require a lot more computing power, and so our approximation should be optimized with respect to that.
The question now is *How did we get such a nice initial guess, which allowed us to construct the sum?* The answer is - we didn't! There's a formal mathematical way to find the terms of such a sum and our chances of actually guessing them are quite slim for a general function! Although we won't go through the math formally, the next section is dedicated to deriving a way for computing the terms and so constructing general waves from any function with any desired period!

3 Fourier series

First note that the cosine function is just a sine function with an additional phase of $\frac{\pi}{2}$ - $\sin(t + \frac{\pi}{2}) = \cos(t)$. In our discussion above, we took the starting phase to be $\varphi_0 = 0$, since our function was odd (like the sine function). An odd function is a function which changes its sign when we change the sign of its variable - as an example $\sin(-x) = -\sin(x)$. An even function, on the other hand, does NOT change its sign when we change the sign of its variable - as an example $\cos(-x) = \cos(x)$. So the cos function is just a sine function, moved by a half-period so it becomes even. To allow for any function to be approximated by our series (both odd, even or neither of the two), we must take both cosines and sines in it.

The process of assigning a series of sine and cosine functions to a general function $f(t)$ for a given period T is referred to as Fourier analysis and the series themselves are called Fourier series. Let us start with the assumption that a general periodic function $f(t)$, having a period T can be obtained as the following sum:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right] \tag{3.1}$$

Where $\{a_n\}$ and $\{b_n\}$ are some real constants, to be determined. Note that since $\cos(0) = 1$ and $\sin(0) = 0$, the first constant a_0 is simply the constant in front of the cosine for $n = 0$, which has been separated from the rest of the sum, and the factor $1/2$ has been added for later convenience.
We will make use the following two results:

$$\int_0^T \sin\left(\frac{2\pi nt}{T}\right) \sin\left(\frac{2\pi mt}{T}\right) dt = \frac{T}{2}\delta_{nm} \tag{3.2}$$

$$\int_0^T \cos\left(\frac{2\pi nt}{T}\right) \cos\left(\frac{2\pi mt}{T}\right) dt = \frac{T}{2}\delta_{nm} \tag{3.3}$$

Where n and m are integers $(n, m = 1, 2, 3, ...)$ and δ_{nm} is the Kronecker symbol, defined as:

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

This tells us, that if we integrate two sine or two cosine functions, for different (integer) multiples of π/T as their coefficients, then the result will be simply 0, and that if we integrate two sine or two cosine functions for the same (integer) multiple of π/T as their coefficient, then the result will be $T/2$! These integrals can be easily computed, using some basic trigonometric identities and table integrals (the reader is encouraged to verify them).
Now let's consider a purely odd function for simplicity. So a function for which $f(-t) = -f(t)$. Then, since the cosines are even, they cannot contribute to this functions's Fourier series - all the coefficients in front of the cosines must be 0 ($a_n = 0$ for $n = 0, 1, 2, ...$). Then expression 3.1 becomes simply:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

Now let's multiply both sides by a different sine. Since we are only summing by the index n, the second sine (say with index m) can enter the sum (as it is a common multiple of each term in it):

$$f(t) \sin\left(\frac{2\pi mt}{T}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right) \sin\left(\frac{2\pi mt}{T}\right)$$

We can now integrate both sides from 0 to T. Formally, the integral and the sum cannot always interchange their positions, but for piecewise continuous functions (functions which are continuous except for finite jumps - like those we are considering), this is not a problem. And so the expression becomes:

$$\int_0^T f(t)\sin\left(\frac{2\pi mt}{T}\right)dt = \sum_{n=1}^{\infty} b_n \int_0^T \sin\left(\frac{2\pi nt}{T}\right)\sin\left(\frac{2\pi mt}{T}\right)dt$$

b_n are constants and so they have been taken out of the integrals. We now have infinitely many integrals on the right hand side, but using 3.2 we can easily see that all of them, except for one, are equal to 0. The only non-zero integral will be the one when $m = n$, and its value will be $T/2$, so the expression becomes:

$$\int_0^T f(t)\sin\left(\frac{2\pi mt}{T}\right)dt = \frac{2}{T}b_m$$

Our aim was to find the constants $\{b_n\}$ in the series 3.1, and now by simply relabelling the index, we see that for a completely odd function, those constants are given by:

$$a_n = 0\ ;\ b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

If we consider a completely even function now, so a function for which $f(-t) = f(t)$, we can use the exact same procedure (but for the cosines) to see that the constants are given by:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)dt\ ;\ b_n = 0$$

Using these two results, we can give the following general formulas for finding the constants for the Fourier expansion of a function $f(t)$ with a period T:

$$a_0 = \frac{2}{T}\int_0^T f(t)dt \tag{3.4}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)dt \tag{3.5}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt \tag{3.6}$$

We can now check these formulas by applying them for the triangular function from the last section. In the region $t \in [0, 2]$, it was given by $f(t) = \frac{1}{2}(t-1)$ and its period was clearly $T = 2$. Since this is obviously an odd function, we can immediately set all $a_n = 0$. In fact, you can try and compute the integrals 3.4 and 3.5 for this function and its period, and they will be 0 as expected. For the coefficients b_n, we have:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt = \frac{1}{2}\int_0^2 (t-1)\sin(\pi nt)dt$$

This integral is easy to compute, using integration by parts, and so:

$$b_n = \frac{1}{2}\left[-\frac{(t-1)}{n\pi}\cos(n\pi t)\Big|_0^2 + \int_0^2 \frac{1}{n\pi}\cos(n\pi t)dt\right] = -\frac{1}{2n\pi}\left[\cos(2n\pi) + \cos(0)\right]$$

Since $\cos(2n\pi) = 1$, this immediately gives us:

$$b_n = -\frac{1}{n\pi}$$

Using this formula, we get:

$$a_1 = -\frac{1}{\pi}\ ;\ a_2 = -\frac{1}{2\pi}\ ;\ a_3 = -\frac{1}{3\pi}; ...$$

which gives us the sum:

$$f(t) = -\frac{1}{\pi}\sin(\pi t) - \frac{1}{2\pi}\sin(2\pi t) - \frac{1}{3\pi}\sin(3\pi t) - ...$$

So guessing is not really required! Given a periodic piecewise continuous function with period T and an expression for it in the region $t \in [0, T]$, we can find a Fourier series for it by computing 3.4, 3.5 and 3.6 and plugging them into 3.1.
Each of the factors inside the sines and cosines is of the form $2\pi n/T$, and since we defined angular frequency ω, given by 1.1, we can see that the Fourier series represent many waves of different frequencies, added together! The expression 3.1 can then be rewritten as:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n sin(\omega_n t)] \tag{3.7}$$

Where the angular frequencies ω_n are given by:

$$\omega_n = n\omega = \frac{2\pi n}{T} \tag{3.8}$$

In this way, we can see the Fourier process as a way of dissolving a general wave into its different components, each of which having a frequency, which is a multiple of the angular frequency ω for the period T. The frequencies ω_n and the coefficients $\{a_n\}$ and $\{b_n\}$ form the "spectre" of a given function, or as it is in most cases, some signal. They completely determine the shape of the signal, and allow us to fully reconstruct it!
We see, that if we had a periodic signal, we can decompose it into its components and find its spectre (as we just did). The opposite is obviously also true - if we have the spectre of a signal (so the constant and the frequencies for each n), we can construct it. This duality between the signal as a function of time (in this case) and the series as a function of the frequencies, goes much deeper as we will see later, when we try to decompose non-periodic signals in a similar way! To conclude with this chapter, let us consider another very traditional example - a rectangular signal.
Example: Find the Fourier series for the function $f(x)$ with period $T = 2$, defined in $x \in [0, 2]$ as follows:

$$f(x) = \begin{cases} c & \text{for } 0 < x < 1 \\ -c & \text{for } 1 < x < 2 \end{cases}$$

As the name suggests, this function is simply a rectangle of sides c and 1 (alternating above and below the x axis). The function is obviously odd again, so we only need to consider the b_n terms. Calculation is straightforward and the coefficients are $b_n = \frac{2c}{n\pi}(1 - (-1)^n)$ (verify this). The plot bellow shows the first N terms of the Fourier series for $N = 1, 5, 9, 13$ and the original function in the case $c = 0.35$.

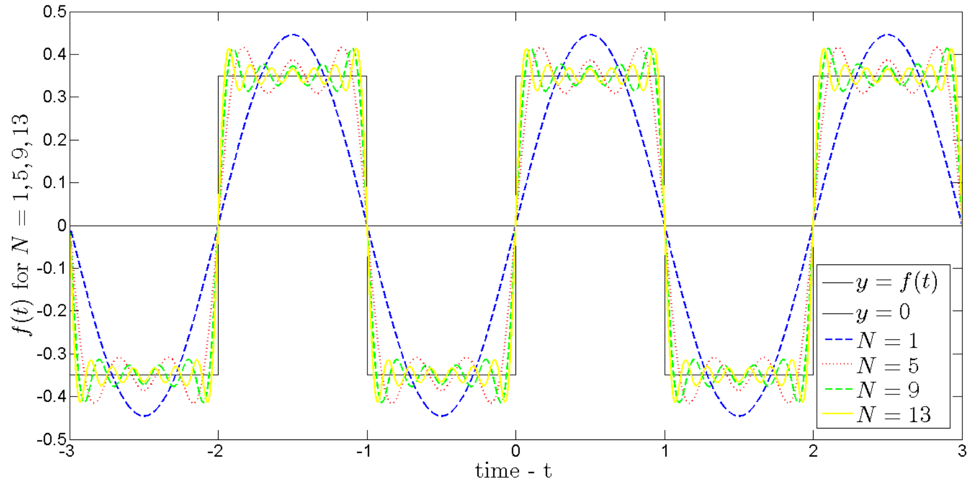

Fig.2.7 Illustration to the example.